Algebra I

A Brief but Comprehensive Study Guide

- Concise lecture notes written from a new perspective
- All-inclusive examples with step-by-step solutions
- Opportunities to further your understanding

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About the Author

Steven C. Silvestri is a student at St. Joseph's College, studying Mathematics with a concentration in Adolescent Education. In 2017, Steven graduated from Eastport South Manor Jr. Sr. High, ranking 4th in his class. It was during his high school years that he recognized his profound appreciation for mathematics and decided that he wished to pursue a career path similar to the teachers that inspired him. Steven currently shares his passion for mathematics as an instructor at Mathnasium. To learn more about the author visit StevenCSilvestri.com.

Dedication

To Mom

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1.1 – Laying the Foundation

In this section, we are going to discuss the general concept of algebra. This will serve to familiarize you with the ideas that we will soon explore. Now, I could speak about algebra in the context of its origin – where it was first used and discovered – but I have a feeling you don't really care about that and I don't blame you! What I do suspect you want to know is:

- What's with all of these letters? This *ain't no* English class...
- Why do I gotta learn this? How is it useful?
- What even is algebra? *Ahhhh!*

What is Algebra?

Algebra is the study of mathematics used to determine the value of unknowns, given pieces of information. These unknowns are typically represented by letters, called *variables*. In contrast, known numerical values are referred to as *constants*. *Algebraic expressions* consist of constants and variables (e.g. $x + 3$ or $2x + 1$). An *algebraic equation* defines relationship between algebraic expressions (e.g. $x + 3 = 2x + 1$).

Constants (e.g.)	Variables (e.g.)	Equality or Inequality Signs
-5	x	$=$
-2	y	$\neq$
0	z	$\leq$
1	p	$<$
π	q	$\geq$
5	r	$>$

Our most basic algebraic equation may look like: $x = 2$. Now there isn't much to solve here; the value of the variable is already known: it is 2. A slightly more complex algebraic equation is: $x + 1 = 2$. Simply from observing, you may notice that the only possible value for our variable x is 1. However, given more complicated equations, the answer may not be so apparent. For instance, take a look at the following equation:

$$\underbrace{\underbrace{2x+1}_{\text{algebraic expression}} = \underbrace{x+3}_{\text{algebraic expression}}}_{\text{algebraic equation}}$$

To determine the value of the variable, we must manipulate the equation so that the variable is isolated. In the next lesson, I will prepare you with the tools to do so. Note, in the algebraic expression to the left of the equal sign, there is a 2 placed before the x. Implied between the 2 and the x is a multiplication sign. We refer to this 2 as the multiplier or more formally, the *coefficient*.

I understand this is a lot to take in. No worries! As we progress further, you will slowly become more acquainted with the terminology and the respective roles they play. However, before you go on, I would like to leave you with the following motivation for studying algebra.

Motivation for Studying Algebra!

- Algebra *will* follow you into higher levels of mathematics such as geometry and statistics.
- You may benefit from algebra in real life – you may use algebra to make better financial decisions, for home improvement projects, or even to balance ingredients when cooking!
- Algebra strengthens your mind; your ability to reason and form logic!

Algebra The study of mathematics used to determine the value of unknowns, given pieces of information.

Algebraic Equation A stated relationship between algebraic expressions ($x + 2 = 2x + 1$).

Algebraic Expression Consists of constants and variables ($x + 2$)

Coefficient The multiplier to a variable (the 2 from $2x$)

Constant A known numerical value that does not change (2).

Variable A placeholder for unknown values; it is assumed to vary in value (x).

1.2 – The Game of Algebra

Solving an algebraic equation is much like a game – and like every game, there are instructions on how to play. Think of this section as your instruction manual and think of each algebraic equation as a puzzle.

Similar to how one performs rotations to a Rubik's cube to achieve the desired form, we are going to manipulate our algebraic equation into a form that reveals the value of our variable. We achieve this by isolating the variable, moving all other terms to the opposite side of equal sign.

How To Play:

Rule 1: You may add, subtract, multiply and divide by any quantity, as long as you do the same to both sides of the equation.

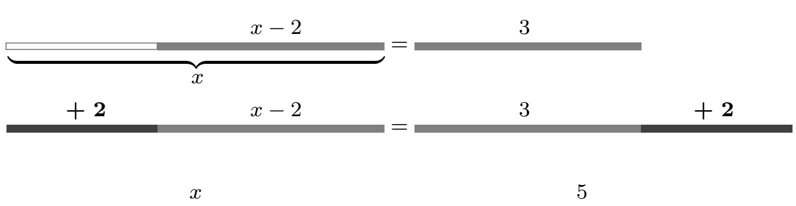
Rule 2: To move a quantity or variable to the other side of the equation, we must perform the opposite operation.

These two rules yield the following:

The Addition Property of Equality

$$\begin{aligned} x - 2 &= 3 \\ \mathbf{+2} &= \mathbf{+2} \\ \\ x &= 5 \end{aligned}$$

$\Downarrow$ *Visual* $\Downarrow$



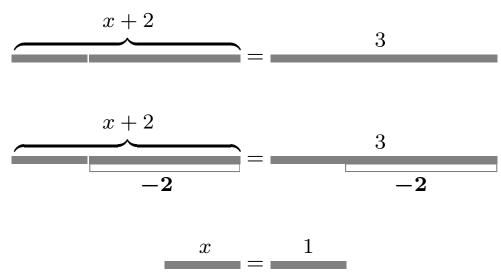
Note: The length of each line segment is drawn with respect to the magnitude of each expression. By following rule 1, we ensure that our lengths remains consistent on both sides of the equal sign; we ensure that our equation remains balanced.

The Subtraction Property of Equality

$$x + 2 = 3$$
$$\mathbf{-2} = \mathbf{-2}$$

$$x = 1$$

$\Downarrow$ *Visual* $\Downarrow$



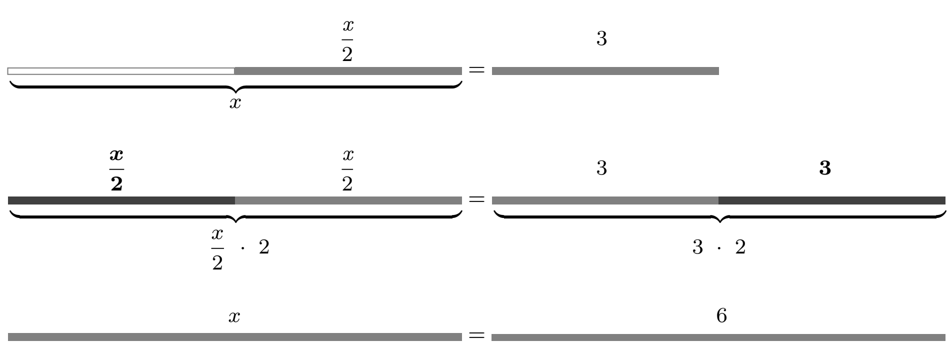
The Multiplication Property of Equality

$$\frac{x}{2} = 3$$

$$\mathbf{2} \cdot \frac{x}{2} = 3 \cdot \mathbf{2}$$

$$x = 6$$

$\Downarrow$ *Visual* $\Downarrow$



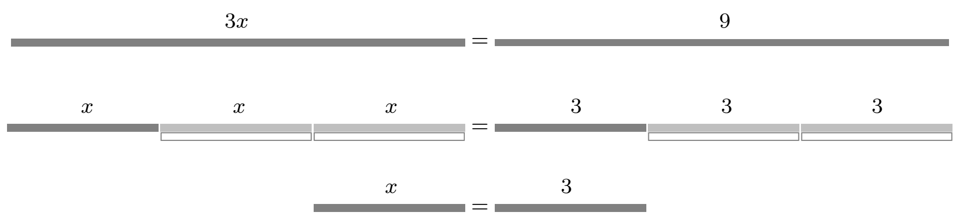
The Division Property of Equality

$$3x = 9$$

$$\frac{3x}{\mathbf{3}} = \frac{9}{\mathbf{3}}$$

$$x = 3$$

$\Downarrow$ *Visual* $\Downarrow$

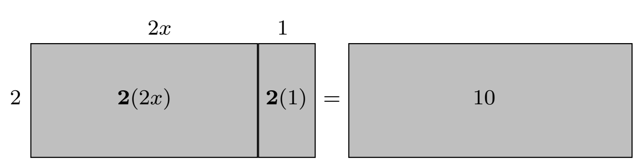


Other useful algebraic properties:

The Distributive Property of Multiplication Over Addition

$$\begin{aligned} 2(2x+1) &= 10 \\ \mathbf{2}(2x) + \mathbf{2}(1) &= 10 \\ 4x + 2 &= 10 \\ &\vdots \\ x &= 2 \end{aligned}$$

$\Downarrow$ *Visual* $\Downarrow$

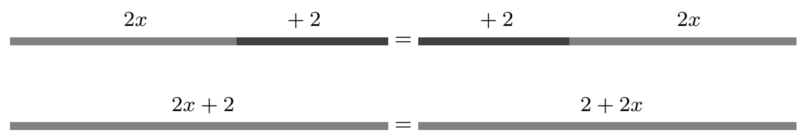


Note: The Distributive Property of Multiplication Over Addition may be illustrated as solving for area. The sum of each part's area is equal to the total area.

The Commutative Property of Addition

$$2x + 2 = 2 + 2x$$

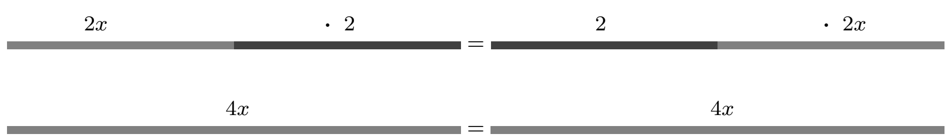
$\Downarrow$ *Visual* $\Downarrow$



The Commutative Property of Multiplication

$$2x \cdot 2 = 2 \cdot 2x$$

$\Downarrow$ *Visual* $\Downarrow$



Cheat Codes:

- Given a fractional coefficient (ex. the $\frac{3}{4}$ in $\frac{3}{4}x$), multiply both sides by the *reciprocal* (ex. $\frac{4}{3}$). The multiplication property of equality and the division property of equality, together, allow this.
- If possible, simplify each expression first.
- Check your work by plugging your value into the original equation.

1.3 – Solving Linear Equations

A wise man once said, "I did it my way." That man's name was Frank Sinatra – the famous Italian-American singer. Now of course he wasn't singing about algebra; he was singing about his self-directed lifestyle. In the same spirit as the words sang by Mr. Sinatra, you have the freedom to choose your own direction when solving linear equations. In this section, we are going to explore some examples that demonstrate this.

Example 1:

Solve the equation $x + 8 = 14$ for x.

Solution: Use the Subtraction Property of Equality to subtract 8 from both sides of the equation.

$$\begin{aligned} x + 8 &= 14 \\ \mathbf{-8} &= \mathbf{-8} \\ \\ x &= 6 \end{aligned}$$

Check Work: The most powerful and widely applicable cheat code in the game of algebra is to check your work. Algebra is a game of diligence and not speed – and so, I advise, before prematurely submitting your work, that you check it first. To check our work in example 1, we substitute our value for x back into the original equation ($x + 8 = 14 \Rightarrow 6 + 8 = 14 \Rightarrow 14 = 14$ ✓).

Example 2a:

Solve the equation $x - 8 = 14$ for x.

Solution: Use the Addition Property of Equality to add 8 to both sides of the equation.

$$\begin{aligned} x - 8 &= 14 \\ \mathbf{+8} &= \mathbf{+8} \\ \\ x &= 22 \end{aligned}$$

Example 2b:

Solve the equation $-8 + x = 14$ for x.

Solution: Use the Addition Property of Equality to add 8 to both sides of the equation.

$$\begin{aligned} -8 + x &= 14 \\ \mathbf{+8} \quad\quad &= \mathbf{+8} \\ \\ x &= 22 \end{aligned}$$

Note, the equations in example 2a and 2b are equivalent. The -8 can go on either side of the variable.

Enrich Your Understanding: Subtraction is the addition of a negative number. Therefore, $x - 8$ can be written as, $x + -8$. The commutative property allows us to swap the order of addition: x $+ -8 = -8 + x$. **Please Be Careful:** $x - 8 \neq 8 - x$.

Example 3:

Solve the equation $\frac{6x}{7} = 9$ for x.

Solution 1: Multiply both sides by the coefficient's reciprocal.

$$\begin{aligned} \frac{6x}{7} &= 9 \\ \frac{\mathbf{7}}{\mathbf{6}} \cdot \frac{6x}{7} &= 9 \cdot \frac{\mathbf{7}}{\mathbf{6}} \\ x &= \frac{63}{6} \\ x &= \frac{21}{2} \end{aligned}$$

Enrich Your Understanding: A fraction represents division. Therefore, $\frac{6x}{7}$ may be read as $6x$ divided by 7. This means we are able to multiply both sides of the equation by 7 (allowed by the Multiplication Property of Equality). $\frac{6x}{7} = 9 \longrightarrow 7 \cdot \frac{6x}{7} = 9 \cdot 7 \longrightarrow 6x = 63$. Now to completely isolate our variable, we must divide both sides by 6 (allowed by the Division

Property of Equality). $\frac{6x}{6} = \frac{63x}{6} \longrightarrow x = \frac{21}{2}$. In short, multiplying by 7 and dividing by 6 can be expressed as $\frac{7}{6}$, which is the reciprocal of $\frac{6}{7}$.

Example 4:

Solve the equation $2x + 4 = 14$ for x.

Solution 1: Step one, use the Subtraction Property of Equality to subtract 4 from both sides of the equation. Step two, use the Division Property of Equality to divide both sides of the equation by 2.

$$\begin{aligned} 2x + 4 &= 14 \\ \mathbf{-4} &= \mathbf{-4} \\[1em] \frac{2x}{\mathbf{2}} &= \frac{10}{\mathbf{2}} \\[1em] x &= 5 \end{aligned}$$

Solution 2: Step one, use the Division Property of Equality to divide both sides of the equation by 2. Step two, use the Subtraction Property of Equality to subtract 2 from both sides of the equation. **Make sure to divide the entire left side by 2**

$$\begin{aligned} 2x + 4 &= 14 \\[1em] \frac{2x + 4}{\mathbf{2}} &= \frac{14}{\mathbf{2}} \\[1em] x + 2 &= 7 \\ \mathbf{-2} &= \mathbf{-2} \\[1em] x &= 5 \end{aligned}$$

Example 5:

Solve the equation $\frac{1}{5}(5x + 10) + 13 = 30$ for x.

Solution 1: Step one, use the Subtraction Property of Equality to subtract 13 from both sides of the equation. Step two, use the Distributive Property of Multiplication over Addition to multiply both terms inside of the parenthesis by $\frac{1}{5}$. Step three, use the Subtraction Property of Equality to subtract 2 from both sides of the equation.

$$\begin{aligned}
\frac{1}{5}(5x+10)+13 &= 30 \\
\mathbf{-13} &= \mathbf{-13} \\[1em]
\frac{1}{5}(5x+10) &= 17 \\[1em]
x+2 &= 17 \\
\mathbf{-2} &= \mathbf{-2} \\[1em]
x &= 15
\end{aligned}$$

Solution 2: Step one, use the Subtraction Property of Equality to subtract 13 from both sides of the equation. Step two, multiply both sides by the reciprocal. Step three, use the Subtraction Property of Equality to subtract 10 from both sides of the equation. Step four, use the Division Property of Equality to divide both sides by 5.

$$\begin{aligned}
\frac{1}{5}(5x+10)+13 &= 30 \\
\mathbf{-13} &= \mathbf{-13} \\[1em]
\mathbf{5}\cdot\frac{1}{5}(5x+10) &= 17\cdot\mathbf{5} \\[1em]
5x+10 &= 85 \\
\mathbf{-10} &= \mathbf{-10} \\[1em]
\frac{5x}{\mathbf{5}} &= \frac{75}{\mathbf{5}} \\[1em]
x &= 15
\end{aligned}$$

Solution 3: Step one, multiply both sides by the reciprocal. Step two, simplify the left expression. Step three, use the Subtraction Property of Equality to subtract 75 from both sides of the equation. Step four, use the Division Property of Equality to divide both sides by 5.

$$\frac{1}{5}(5x + 10) + 13 = 30$$

$$\mathbf{5} \cdot [\frac{1}{5}(5x + 10) + 13] = 30 \cdot \mathbf{5}$$

$$\mathbf{5} \cdot \frac{1}{5}(5x + 10) + \mathbf{5}(13) = 30 \cdot \mathbf{5}$$

$$5x + 10 + 65 = 150$$

$$\begin{aligned} 5x + 75 &= 150 \\ \mathbf{-75} &= \mathbf{-75} \end{aligned}$$

$$\frac{5x}{\mathbf{5}} = \frac{75}{\mathbf{5}}$$

$$x = 15$$

Enrich Your Understanding: Notice in solution 3, on our second line, we have square brackets within our expression. Square brackets are synonymous to parentheses. Square brackets and parentheses, alike, imply that we use the Distributive Property of Multiplication Over Addition.

Summary:

There exist multiple ways to solve linear equations. This allows you to solve it *your* way. Still, you must follow the basic laws of algebra, as covered in section 1.2. Below is a reiteration of these laws.

- You may add, subtract, multiply and divide by any quantity, as long as you do the same to **both sides** of the equation
- To move a quantity or variable to the other side of the equation, we must perform the **opposite operation**.
- These two rules yield the following: the Addition Property of Equality, the Subtraction Property of Equality, the Multiplication Property of Equality, and the Division Property of Equality.

1.4 – Solving Literal Equations

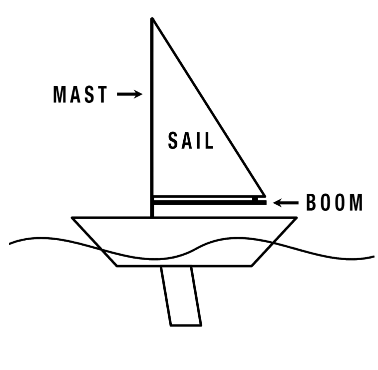
WHAT IS A LITERAL EQUATION? A *literal equation* is an equation with multiple variables. Mathematical formulas are a perfect example of this. For instance, the area A of a triangle is defined by the formula $A = \frac{1}{2} \cdot b \cdot h$ where b is the base and h is the height. Suppose we wish to define the height of the triangle. This is achieved by isolating our variable h, in the same manner that A is isolated to define area. We achieve this by following the same rules that we used to solve linear equations. The key to solving literal equations is to handle our other variables as we would handle constants and coefficients.

Literal Equations vs. Linear Equations

Literal Equation	*Linear Equation*
$A = \frac{1}{2} \cdot b \cdot h$	$6 = \frac{1}{2} \cdot b \cdot 3$
$y = mx + b$	$2 = 3x + 9$
$\Sigma = n_1 + n_2 + n_3 + n_4$	$120 = 24 + 40 + n_3 + 26$

Example 1:

A sail boat manufacturing company needs to determine the mast height for a new boat model. The area of the sail is 100 sq. ft and the length of the boom is 10 ft. Jessica, the hired mathematician, is asked to calculate the height of the mast. Jessica accomplishes this by using the formula $A = \frac{1}{2} \cdot b \cdot h$. She substitutes her known values into the formula and then solves for h. Jessica is then asked to repeat this task for three other concept models. Working smarter and not harder, Jessica realizes that she could solve for h first, prior to substituting. This would allow her to perform the algebra once, providing her with a rearranged equation that requires purely basic arithmetic! Help Jessica solve for h.



Solution: We can begin to isolate our variable h by applying the Multiplication Property of Equality to eliminate the 2 from the right side. Next, we must eliminate the b from the Literal Equation in the same way that we eliminate the 6 from the Linear Equation. Handling the b as the coefficient to our variable h, we apply the Division Property of Equality.

Linear Equation	*Literal Equation*	*Properties*
$100 = \frac{1}{2} \cdot 10 \cdot h$	$A = \frac{1}{2} \cdot b \cdot h$	
$\mathbf{2} \cdot 100 = \frac{1}{2} \cdot 10 \cdot h \cdot \mathbf{2}$	$\mathbf{2} \cdot A = \frac{1}{2} \cdot b \cdot h \cdot \mathbf{2}$	(i.) The Multiplication Property of Equality
$\frac{200}{\mathbf{10}} = \frac{10}{\mathbf{10}} \cdot h$	$\frac{2A}{\boldsymbol{b}} = \frac{b}{\boldsymbol{b}} \cdot h$	(ii.) The Division Property of Equality
$h = 10$	$h = \frac{2A}{b}$	

Enrich Your Understanding: Recall, the reason that we solve for our variable h is so that we may define height, in terms of base and area. This allows for others to use your rearranged formula to determine height given the values of base and area. In other words, you performed the algebra for them, so they don't have to!

Example 2:

Social media engagement rate is calculated by the equation $E = \frac{l\ +\ c}{f} \cdot 100\%$ where E is the engagement rate, l is the amount of likes, c is the amount of comments, and f is the amount of followers. Solve for f.

Solution: We are able to isolate our variable f using the Multiplication Property of Equality to bring our variable to left side of the equation. Lastly, since we now have $f \cdot E$ on the left side, we must use the Division Property of Equality to fully isolate our variable f.

Literal Equation	*Properties*
$E = \dfrac{l\ +\ c}{f} \cdot 100\%$	
$\boldsymbol{f} \cdot E = \dfrac{l\ +\ c}{f} \cdot 100\% \cdot \boldsymbol{f}$	(i.) The Multiplication Property of Equality
$\dfrac{f \cdot E}{\boldsymbol{E}} = \dfrac{(l\ +\ c) \cdot 100\%}{\boldsymbol{E}}$	(ii.) The Division Property of Equality
$f = \dfrac{(l\ +\ c) \cdot 100\%}{E}$	

Enrich Your Understanding: Notice in this example that I brought the variable we were solving for to the other side of the equation. Sometimes it is simpler to solve for our variable by eliminating it, rather than eliminating all of the other terms surrounding it.

Example 3:

The surface area of a rectangle is defined by $S = 2(LW + HW + HL)$ where S is the surface area, L is the length, W is the width, and H is the height. Solve for L.

Solution: Our goal is to isolate our variable L. To begin this, we apply the Division Property of Equality to divide both sides by 2. Note that a quantity beside a parenthesis and a quantity beside a variable both imply multiplication. Step ii., we apply the Subtraction Property of Equality to subtract HW from both sides. Step iii., we perform the reverse of the Distributive Property of Multiplication Over Addition to collect L from LW and HL. Step iv., we apply the Division Property of Equality to divide both sides by $H + W$.

Literal Equation	*Properties*
$S = 2(LW + HW + HL)$	
$\dfrac{S}{\mathbf{2}} = \dfrac{2(LW + HW + HL)}{\mathbf{2}}$	(i.) The Division Property of Equality
$\dfrac{S}{2} = LW + HW + HL$ $\boldsymbol{-HW} = \quad \boldsymbol{-HW}$	(ii.) The Subtraction Property of Equality
$\dfrac{S}{2} - HW = LW + HL$	
$\dfrac{S}{2} - HW = \boldsymbol{L}(W + H)$	(iii.) The Distributive Property of Multiplication Over Addition IN REVERSE
$\dfrac{\frac{S}{2} - HW}{(\boldsymbol{W + H})} = \dfrac{L(W + H)}{(\boldsymbol{W + H})}$	(iv.) The Division Property of Equality
$L = \dfrac{\frac{S}{2} - HW}{(W + H)}$	

Enrich Your Understanding: In this problem, we use the Distributive Property of Multiplication Over Addition in reverse. Another name for this is *factoring*. Take a look at 35 + 5. Our Greatest Common Factor is 5. $35 + 5 = (7 \cdot 5 + 1 \cdot 5)$. Therefore, we may factor out a 5 to obtain the expression, $5(7 + 1)$. Now, take a look at $LW + HL$. In common to both of these terms is the variable L. Therefore, we may factor out the L to obtain the expression, $L(W + H)$.

2.1 – Classifying Polynomials

In the previous chapter, you learned how to solve linear equations using the properties of algebra. I commemorate your effort! In this chapter, we are going to work together to deepen your understanding by dedicating our focus on algebraic expressions.

$$\underbrace{\underbrace{2x+1}_{\text{algebraic expression}} = \underbrace{x+3}_{\text{algebraic expression}}}_{\text{algebraic equation}}$$

English 101

Algebra and English have more in common than you may think! In the English language we have sentences. Inside our sentences we have phrases and inside our phrases we have words. Similarly, in algebra we have equations. Inside our equations we have expressions and inside our expressions we have terms. Sentences = Equations. Phrases = Expressions. Words = Terms. In the same way that words are separated by spaces, terms are separated by addition or subtraction. Terms are made up of constants and/or variables.

$$\underbrace{\overbrace{2x}^{\text{term}} + \overbrace{1}^{\text{term}}}_{\text{algebraic expression}}$$

Naming Expressions

Algebraic expressions may be classified according to the number of terms they have. An algebraic expression with a single term is called a *monomial*. An algebraic expression with two terms is called a *binomial*. An algebraic expression with three terms is called a *trinomial*. Monomials, binomials and trinomials are all kinds of *polynomials*.

Polynomial

Monomial   *Binomial*   *Trinomial*

Note: The prefix *mono-* means one, the prefix *bi-* means two, and the prefix *tri-* means three.

Definition A *polynomial* is a sum of terms. An expression is not a polynomial if it has a negative exponent or if it is has division by a variable (ex. x^{-2} and $\frac{x+1}{x}$ are not polynomials)

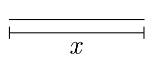
Polynomials are also classified according to their *degree.* The degree is the largest exponent belonging to the variable within the polynomial. Examine the table below:

	1^{st} ***degree***	2^{nd} ***degree***	3^{rd} ***degree***
Monomial	x	x^2	x^3
Binomial	$x+1$	x^2+x	x^3+x^2
Trinomial	–	x^2+x+1	x^3+x^2+x

Another name for a 1^{st} degree polynomial is a *linear* polynomial. Notice in the previous chapter, when solving for *linear* equations, all of our expressions were 1^{st} degree polynomials. Another name for a 2^{nd} degree polynomial is a *quadratic* polynomial. Another name for a 3^{rd} degree polynomial is a *cubic* polynomial.

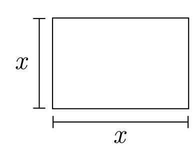
Visualizing Different Degrees Geometrically

Linear
(1^{st} degree)



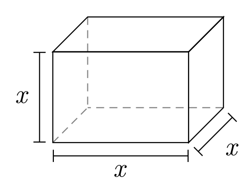
width
$x = x^1$

Quadratic
(2^{nd} degree)



length · width
$x \cdot x = x^2$

Cubic
(3^{rd} degree)



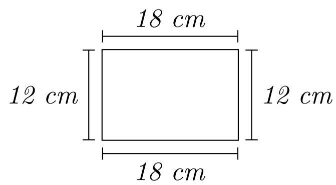
length · width · height
$x \cdot x \cdot x = x^3$

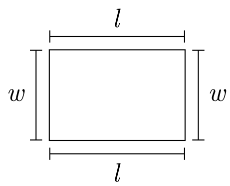
Constant
(0^{th} degree)

•

$x^0 = 1$

2.2 – Combining Like Terms





The perimeter of the rectangle to the left is: 18cm + 12cm + 18cm + 12cm = 60cm. Equivalently, this may be computed as 2(18cm) + 2(12cm) = 60cm.

Likewise, the perimeter of a rectangle may be expressed algebraically as $l + w + l + w$ or $2l + 2w$. The latter expression, $2l + 2w$, is obtained by adding the coefficient of the terms with the same variable component. This is called *combining like terms.* Note that 1 is implied to be the coefficient of a variable without an adjoining constant.

Let's Practice:

Example 1:

Simplify the expression: $2x + x$

$$2x + x \Longrightarrow \underbrace{2x + 1x}_{3x} \Longrightarrow 3x$$

Solution: Since both terms have identical variable components, we may combine them. Recall, when the coefficient is not indicated, it is implied to be 1. Therefore, as 2 + 1 = 3, $2x + x = 3x$.

Example 2:

Simplify the expression: $2x - x$

$$2x - x \Longrightarrow \underbrace{2x - 1x}_{1x} \Longrightarrow x$$

Solution: Since both terms have identical variable components, we may combine them. Therefore, as 2 - 1 = 3, $2x - x = x$.

Example 3:

Simplify the expression: $2x + x + y$

$$2x + x + y \Longrightarrow \underbrace{2x + 1x}_{3x} + y \Longrightarrow 3x + y$$

Solution: As we did in example 1, we may combine $2x + x$ to become $3x$. Since $3x$ and y are not alike, we cannot combine them.

Example 4:

Simplify the expression: $2x + x^2 + y$

$$2x + x^2 + y$$

Solution: We cannot simplify the expression any further. No terms are alike. For terms to be alike, their *entire* variable components must be the same. This includes the variable's exponent.

Example 5:

Simplify the expression: $x^2 + xy + yx + y^2$

$$x^2 + xy + yx + y^2 \Longrightarrow x^2 + \underbrace{xy + yx}_{2xy} + y^2 \Longrightarrow x^2 + 2xy + y^2$$

Solution: According to the commutative property of multiplication, the order of variables within a term does not matter when determining like terms. In other words, we may rewrite $xy + yx$ as $xy + xy$. Since these two terms are alike, we may combine them to become $2xy$.

Example 6:

Simplify the expression: $x^3 + x^2y + xy^2 + y^3 + 2x^2y + 2xy^2 + 2x^2y$

$$x^3 + x^2y + xy^2 + y^3 + 2x^2y + 2xy^2 + 2x^2y$$
$$\Downarrow$$
$$x^3 + \underbrace{(x^2y + 2x^2y + 2x^2y)}_{5x^2y} + \underbrace{(xy^2 + 2xy^2)}_{3xy^2} + y^3$$
$$\Downarrow$$
$$x^3 + 5x^2y + 3xy^2 + y^3$$

Solution: The easiest way to tackle problems with several terms is to begin by grouping. One way to do this is shown in step 2. We group our x^2y terms in one set of parentheses and we group our xy^2 terms in another set of parentheses. Another way to group is to draw certain shapes around alike terms. For example, you may draw circles around our x^2y terms and draw squares around our xy^2 terms. Once grouped, combining like terms becomes much simpler.

Example 7:

Simplify the expression: $x^3 - x^2y + xy^2 - y^3 + 2x^2y - 2xy^2 + 2x^2y$

$$x^3 - x^2y + xy^2 - y^3 + 2x^2y - 2xy^2 + 2x^2y$$
$$\Downarrow$$
$$x^3 + \underbrace{(-x^2y + 2x^2y + 2x^2y)}_{3x^2y} + \underbrace{(xy^2 - 2xy^2)}_{-xy^2} - y^3$$
$$\Downarrow$$
$$x^3 + 3x^2y - xy^2 - y^3$$

Solution: Like the previous example, we start by grouping. When grouping, it is very important to make sure that you do not lose the sign that comes before each term. Notice in step two, when grouping, the minus sign follows each term that precedes it. For example, the x^2y term keeps the minus sign as a negative sign.

Adding and Subtracting Polynomials:

When adding polynomials together, you may dismiss the parentheses and then proceed to combine like terms. When subtracting polynomials, however, it is critical that you distribute the negative sign to each term inside the parentheses of the polynomial that follows.

Example 8:

Simplify the expression: $(x^2 - 8x + 16) + (x^2 - 12x + 36)$

$$
\begin{gathered}
(x^2 - 8x + 16) + (x^2 - 12x + 36) \\
\Downarrow \\
x^2 - 8x + 16 + x^2 - 12x + 36 \\
\Downarrow \\
\underbrace{(x^2 + x^2)}_{2x^2} + \underbrace{(-8x - 12x)}_{-20x} + \underbrace{(16 + 36)}_{52} \\
\Downarrow \\
2x^2 - 20x + 52
\end{gathered}
$$

Solution: Since it is addition, we may scrap the parentheses that encase the two trinomials. Next, we may carefully group like terms, preserving each term's sign. Now we may combine like term.

Example 9:

Simplify the expression: $(x^2 - 8x + 16) - (x^2 - 12x + 36)$

$$
\begin{gathered}
(x^2 - 8x + 16) - (x^2 - 12x + 36) \\
\Downarrow \\
x^2 - 8x + 16 \; - \; x^2 \; + \; 12x \; - \; 36 \\
\Downarrow \\
\underbrace{(x^2 - x^2)}_{0} + \underbrace{(-8x + 12x)}_{4x} + \underbrace{(16 - 36)}_{-20} \\
\Downarrow \\
4x - 20
\end{gathered}
$$

Solution: Since it is subtraction, we must distribute the subtraction sign through the parentheses before combining like terms. Think of $-(x^2 - 12x + 36)$ as $-1(x^2 - 12x + 36)$. Using the Distributive Property of Multiplication Over Addition, we obtain $-x^2 + 12x - 36$. Now we group and combine like terms.

2.3 – Monomial Arithmetic

Previously, we discussed the addition and subtraction of polynomials. This was referred to as combining like terms. In this section, we will conclude monomial arithmetic. We will discuss how to multiply and divide monomials as well as how to simplify monomials raised to an exponent or contained within a radical.

Multiplying Monomials

An exponent denotes the number of times that a value is multiplied by itself (ex. $2^2 = 2 \cdot 2$ and $x^2 = x \cdot x$). When a value is multiplied by itself, we are essentially adding their exponents. For example, $x^2 \cdot x^3 = x^{(2+3)} = x^5$. To confirm this, $x^2 \cdot x^3$ may be expanded as $(x \cdot x) \cdot (x \cdot x \cdot x)$. See this summarized below:

The Product of Powers Property	$\boldsymbol{a^m \cdot a^n = a^{m+n}}$ $\underbrace{2^2}_{2 \cdot 2} \cdot \underbrace{2^3}_{2 \cdot 2 \cdot 2} = \underbrace{2^5}_{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$ $\underbrace{x^2}_{x \cdot x} \cdot \underbrace{x^3}_{x \cdot x \cdot x} = \underbrace{x^5}_{x \cdot x \cdot x \cdot x \cdot x}$

When we multiply monomials that have coefficients, we must multiply the coefficients together and then use the Product of Powers Property to multiply same-*base* variables together. For the term x^2, the *base* is x.

Example 1:

Simplify the expression: $10x \cdot 5x^2 \cdot 2x^5$

$$10x \cdot 5x^2 \cdot 2x^5$$
$$\Downarrow$$
$$(10 \cdot 5 \cdot 2) \cdot x^{(1+2+5)}$$
$$\Downarrow$$
$$100x^8$$

Solution: First, we multiply the coefficients together $(10 \cdot 5 \cdot 2)$. Next, we apply the Product of Powers Property to multiply variables with the same-base together $(x \cdot x^2 \cdot x^5 = x^8)$. Note that when a variable's exponent is not denoted, it is inferred to be 1.

Example 2:

Simplify the expression: $10xy \cdot 5y^2 \cdot 2x^5$

$$10xy \cdot 5y^2 \cdot 2x^5$$
$$\Downarrow$$
$$(10 \cdot 5 \cdot 2) \cdot x^{(1+5)} \cdot y^{(1+2)}$$
$$\Downarrow$$
$$100x^6y^3$$

Solution: First, we multiply the coefficients together $(10 \cdot 5 \cdot 2)$. Next, we apply the Product of Powers Property to multiply variables with the same-base together. ($x \cdot x^5 = x^6$ and $y \cdot y^2 = y^3$).

Dividing Monomials

The opposite of multiplication is division – and so, as you may have guessed, when we divide a value by itself, we subtract their exponents. For example, $x^5 \div x^3 = x^{(5-3)} = x^2$. To confirm this, we may express division as a fraction and then simplify it. See this summarized below:

The Quotient of Powers Property	$\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$ $\frac{2^5}{2^3} = \frac{2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}} = 2^2$ $\frac{x^5}{x^3} = \frac{x \cdot x \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = x^2$

When we divide monomials that have coefficients, we must divide the coefficients and then use the Quotient of Powers Property to divide same-base variables.

Enrich Your Understanding: Division may also be expressed by a negative exponent (ex. x^{-n}). The negative exponent indicates the reciprocal of the positive exponent expression (ex. $x^{-n} = \frac{1}{x^n}$). This demonstrates how the Quotient of Powers Property is obtained from the Product of Powers Property (ex. $x^{-n} \cdot x^n = x^{(-n+n)} = x^0 = 1$ **or** $\frac{x^n}{x^n} = x^{(n-n)} = x^0 = 1$). Recall that subtraction is the addition of a negative value.

Example 3:

Simplify the expression: $20x^6 \div 5x^2$

$$\frac{20x^6}{5x^2} \Longrightarrow \frac{\cancelto{4}{20}}{\cancel{5}} \cdot \frac{x^6}{x^2} \Longrightarrow 4 \cdot x^{(6-2)} \Longrightarrow 4x^4$$

Solution: First, we divide the coefficients ($20 \div 5 = 4$). Next, we apply the Quotient of Powers Property to divide variables with the same-base ($x^6 \div x^2 = x^4$).

Example 4:

Simplify the expression: $18x^9y^3 \div 3x^3y^3z$

$$\frac{18x^9y^3}{3x^3y^3z} \Longrightarrow \frac{\cancelto{6}{18}}{\cancel{3}} \cdot \frac{x^9y^3}{x^3y^3z} \Longrightarrow 6 \cdot x^{(9-3)} \cdot y^{(3-3)} \cdot z^{-1} \Longrightarrow 6x^6z^{-1}$$

Solution: First, we divide the coefficients ($18 \div 3 = 6$). Next, we apply the Quotient of Powers Property to divide variables with the same-base ($x^9 \div x^3 = x^6$ and $y^3 \div y^3 = 1$).

Exponentiating Monomials

When we have an exponent raised to another exponent, we multiply them. See this below:

The Power of a Power Property	$\boldsymbol{(a^m)^n = a^{m \cdot n}}$ $(2^4)^3 = 2^{4\cdot 3} = 2^{12}$ $(x^4)^3 = x^{4\cdot 3} = x^{12}$

When the exponent is on the outside of the parentheses, we must distribute the exponent and perform the Power of a Power Property to each value within.

Enrich Your Understanding: The Power of a Power Property is an extension of the Product of Powers Property (ex. $(x^4)^3 = x^4 \cdot x^4 \cdot x^4 = x^{(4+4+4)} = x^{12}$ **or** $(x^4)^3 = x^{(4\cdot 3)}$).

Example 5:

Simplify the expression: $(2x^2 \cdot 2x^3)^3$

$$(2x^2 \cdot 2y^3)^3 \Longrightarrow 2^3(x^2)^3 \cdot 2^3(y^3)^3 \Longrightarrow 8x^{(2\cdot3)} \cdot 8y^{(3\cdot3)} \Longrightarrow 8x^6 \cdot 8y^9 \Longrightarrow 64x^6y^9$$

Solution: First we distribute our exponent to each value within the parentheses. Specifically, we cube each coefficient and we cube each variable. Next, we apply to the Power of a Power Property to our variables. Lastly, we are able to multiply our constants to fully simplify the expression.

Simplifying Monomial Radicals

As division is the opposite of multiplication, we may think of a radical as the opposite of an exponent. When we have an exponent raised to another exponent, we multiply them. However, when we have the radical of a value, we divide the value's *exponent*, or power, by the radical's *index*, or root. As shown below: n is the root and m is the power. To remember the order that we divide, think of Power Over Root (m/n). Street power lines run above the roots in the ground. When simplifying a radical expression, it is important that we apply Power Over Root to each value within.

- -

Power Over Root

$$\boldsymbol{\sqrt[n]{a^m} = a^{m/n}}$$

$$\sqrt[3]{2^6} = 2^{(\frac{6}{3})} = 2^2 = 4$$

$$\sqrt[3]{x^6} = x^{(\frac{6}{3})} = x^2$$

- -

Enrich Your Understanding: The root of a radical may also be expressed as an exponent (ex. $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}}$). This demonstrates how Power Over Root is obtained from the Product of Powers Property (ex. $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{(m\cdot\frac{1}{n})} = a^{(\frac{m}{n})}$).

Example 6:

Simplify the expression: $\sqrt[3]{27x^{12}y^3}$

$$\sqrt[3]{27x^{12}y^3} \Longrightarrow 27^{(\frac{1}{3})}x^{(\frac{12}{3})}y^{(\frac{3}{3})} \Longrightarrow (3^3)^{\frac{1}{3}}x^{(\frac{12}{3})}y^{(\frac{3}{3})} \Longrightarrow 3^{(\frac{3}{3})}x^{(\frac{12}{3})}y^{(\frac{3}{3})} \Longrightarrow 3x^4y$$

Solution: To simplify this expression, we must take the cube root of each value within the radical by performing Power Over Root. The cube root of 27 is 3.

2.4 – A Polynomial Multiplied by a Monomial

To multiply a polynomial by a monomial, we must enclose our polynomial in parentheses and apply the Distributive Property of Multiplication Over Addition. This is very much the same as multiplying a polynomial by a constant. Below is a screenshot from section 1.2.

The Distributive Property of Multiplication Over Addition

$$\begin{aligned} 2(2x+1) &= 10 \\ \mathbf{2}(2x)+\mathbf{2}(1) &= 10 \\ 4x+2 &= 10 \\ &\vdots \\ x &= 2 \end{aligned}$$

Example 1:

Multiply $4x^2 + 3x$ by $3x^3$.

$$3x^3(4x^2 + 3x) \Longrightarrow 3x^3 \cdot 4x^2 + 3x^3 \cdot 3x \Longrightarrow 12x^5 + 9x^4$$

Solution: First, we enclose our binomial $4x^2 + 3x$ in parentheses. Next, we distribute our monomial $3x^3$ through the parentheses. Now, we perform simple monomial multiplication. $3x^3 \cdot 4x^2 = 12x^{(2+3)} = 12x^5$ and $3x^3 \cdot 3x = 9x^{(1+3)} = 9x^4$.

Example 2:

Multiply $-4x$ by $2x^3 + 3x^2$.

$$-4x(2x^3 + 3x^2) \Longrightarrow (-4x \cdot 2x^3) + (-4x \cdot 3x^2) \Longrightarrow -8x^4 - 12x^3$$

Solution: First, we enclose our binomial $2x^3 + 3x^2$ in parentheses. Next, we distribute our monomial $-4x$ through the parentheses. It is important that you distribute the entire term, including the negative sign. Now, we perform simple monomial multiplication. $-4x \cdot 2x^3 = -8x^{(1+3)} = -8x^4$ and $-4x \cdot 3x^2 = -12x^{(1+2)} = -12x^3$. Since our second product is negative $(-12x^3)$, we may write our expression as subtraction.

Example 3:

Multiply $3x^2 - 2x + 10$ by $-y^2$.

$$-y^2(3x^2 - 2x + 10)$$

$$\Downarrow$$

$$(-y^2 \cdot 3x^2) + (-y^2 \cdot -2x) + (-y^2 \cdot 10)$$

$$\Downarrow$$

$$-3x^2y^2 + 2xy^2 - 10y^2$$

Solution: First, we enclose our trinomial $3x^2 - 2x + 10$ in parentheses. Next, we distribute our monomial $-y^2$ through the parentheses. It is important that you distribute the entire term, including the negative sign. Now, we perform simple monomial multiplication. Remember that we only add the exponents of the same variable when multiplying.

Example 4:

Multiply $3x^2 - 2xy + 3y$ by xy^2.

$$xy^2(3x^2 - 2xy + 3y)$$

$$\Downarrow$$

$$(xy^2 \cdot 3x^2) + (xy^2 \cdot -2xy) + (xy^2 \cdot 3y)$$

$$\Downarrow$$

$$3x^3y^2 - 2x^2y^3 + 3xy^3$$

Solution: First, we enclose our trinomial $3x^2 - 2xy + 3y$ in parentheses. Next, we distribute our monomial xy^2 through the parentheses. Now, we perform simple monomial multiplication. Remember that we only add the exponents of the same variable when multiplying.

2.5 – A Polynomial Multiplied by a Polynomial

Below is an algebraic expression that prompts polynomial by polynomial multiplication:

$$(x + 2) \cdot (x^2 + 2x)$$

If your current thoughts are: "how the *heck* am I going to do this?!," I promise you that this section is no more difficult than the previous. In fact, it is pretty much the same – and with a new trick, you might actually find it easier!

Repeated Distribution

Repeated Distribution is a textbook approach for carrying out polynomial by polynomial multiplication. This method suggests that we repeat the Distributive Property of Multiplication Over Addition for each term of our 1^{st} factor.

Example 1:

Multiply $x + 2$ by $x^2 + 2x$.

$$(\boldsymbol{x} \boldsymbol{+} \mathbf{2})(x^2 + 2x)$$

$$\boldsymbol{x}(x^2 + 2x) \boldsymbol{+} \mathbf{2}(x^2 + 2x)$$

$$\Downarrow$$

$$(\boldsymbol{x} \cdot x^2 + \boldsymbol{x} \cdot 2x) \boldsymbol{+} (\mathbf{2} \cdot x^2 + \mathbf{2} \cdot 2x)$$

$$\Downarrow$$

$$x^3 + 2x^2 + 2x^2 + 4x$$

$$\Downarrow$$

$$x^3 + 4x^2 + 4x$$

Solution: First, we enclose both binomials in parentheses. Now, we distribute the 1^{st} term of our 1^{st} factor through our 2^{nd} factor $\rightarrow x(x^2 + 2x)$, and we distribute the 2^{nd} term of our 1^{st} factor through our 2^{nd} factor $\rightarrow 2(x^2 + 2x)$. Once distributed, we combine like terms.

Example 2:

Multiply $x-2$ by x^2-2x+1.

$$(\boldsymbol{x}-\mathbf{2})(x^2-2x+1)$$

$$\boldsymbol{x}(x^2-2x+1)-\mathbf{2}(x^2-2x+1)$$

$$\Downarrow$$

$$(\boldsymbol{x}\cdot x^2+\boldsymbol{x}\cdot -2x+\boldsymbol{x}\cdot 1)+(-\mathbf{2}\cdot x^2+-\mathbf{2}\cdot -2x+-\mathbf{2}\cdot 1)$$

$$\Downarrow$$

$$x^3-2x^2+x-2x^2+4x-2$$

$$\Downarrow$$

$$x^3-4x^2+5x-2$$

Solution: First, we enclose both binomials in parentheses. Now, we distribute the 1^{st} term of our 1^{st} factor through our 2^{nd} factor $\rightarrow x(x^2-2x+1)$, and we distribute the 2^{nd} term of our 1^{st} factor through our 2^{nd} factor $\rightarrow -2(x^2-2x+1)$. Note, we must not ignore the subtraction sign. The subtraction sign is included as a negative sign in our 2^{nd} term. When distributing, be sure to distribute the entire term. Once distributed, we combine like terms.

Example 3:

Multiply x^2-4x+2 by x^2-2x+1.

$$(\boldsymbol{x^2}-\mathbf{4}\boldsymbol{x}+\mathbf{2})(x^2-2x+1)$$

$$\boldsymbol{x}(x^2-2x+1)-\mathbf{4}\boldsymbol{x}(x^2-2x+1)+\mathbf{2}(x^2-2x+1)$$

Solution: As done in previous examples, we must distribute each term of our 1^{st} factor through our 2^{nd} factor. Since our 1^{st} factor is a trinomial, we must apply the Distribution of Multiplication Over Addition Property a total of 3 times.

As you can see with this example, Repeated Distribution starts to become overwhelming, prone for simple mistakes. Let's look at another method for polynomial by polynomial multiplication.

The Box Method

The *Box Method* provides a more organized layout for Repeated Distribution. This allows us to visualize which terms we must multiply together. To demonstrate the Box Method, let us take a second look at our examples.

Example 4:

Multiply $x + 2$ by $x^2 + 2x$.

$$\begin{array}{|c|c|c|} \hline & x^2 & 2x \\ \hline x & & \\ \hline 2 & & \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|c|} \hline & x^2 & 2x \\ \hline x & \boldsymbol{x^3} & \boldsymbol{2x^2} \\ \hline 2 & \boldsymbol{2x^2} & \boldsymbol{4x} \\ \hline \end{array} \Longrightarrow x^3+2x^2+2x^2+4x \Longrightarrow x^3+4x^2+4x$$

Solution: First, we set up a grid for binomial by binomial multiplication. Now, we label our rows according to the terms within our 1^{st} factor and our columns according to the terms within our 2^{nd} factor. Next, we multiply each row by each column. Lastly, we add our products together and combine like terms.

Example 5:

Multiply $x - 2$ by $x^2 - 2x + 1$.

$$\begin{array}{|c|c|c|c|} \hline & x^2 & -2x & 1 \\ \hline x & & & \\ \hline -2 & & & \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|c|c|} \hline & x^2 & -2x & 1 \\ \hline x & \boldsymbol{x^3} & \boldsymbol{-2x^2} & \boldsymbol{x} \\ \hline -2 & \boldsymbol{-2x^2} & \boldsymbol{4x} & \textbf{-2} \\ \hline \end{array} \Longrightarrow$$

$$x^3 - 2x^2 + x - 2x^2 + 4x - 2 \Longrightarrow x^3 - 4x^2 + 5x - 2$$

Solution: First, we set up a grid for binomial by trinomial multiplication. Now, we label our rows according to the terms within our 1^{st} factor and our columns according to the terms within our 2^{nd} factor. Next, we multiply each row by each column. Lastly, we add our products together and combine like terms.

Example 6:

Multiply $x^2 - 4x + 2$ by $x^2 - 2x + 1$.

	x^2	$-2x$	1
x^2			
$-4x$			
2			

$\Longrightarrow$

	x^2	$-2x$	1
x^2	$\boldsymbol{x^4}$	$\boldsymbol{-2x^3}$	$\boldsymbol{x^2}$
$-4x$	$\boldsymbol{-4x^3}$	$\boldsymbol{8x^2}$	$\boldsymbol{-4x}$
2	$\boldsymbol{2x^2}$	$\boldsymbol{-4x}$	$\boldsymbol{2}$

$\Longrightarrow$

$$x^4 - 2x^3 + x^2 - 4x^3 + 8x^2 - 4x + 2x^2 - 4x + 2 \Longrightarrow x^4 - 6x^3 + 11x^2 - 8x + 2$$

Solution: First, we set up a grid for trinomial by trinomial multiplication. Now, we label our rows according to the terms within our 1^{st} factor and our columns according to the terms within our 2^{nd} factor. Next, we multiply each row by each column. Lastly, we add our products together and combine like terms.

3.1 – Greatest Common Factor

Factors are values that when multiplied together yield an expected value. For example, given an expected value of 6, our set of factors would be $\{1, 2, 3, 6\}$, since $1 \cdot 6 = 6$ and $2 \cdot 3 = 6$.

Now, suppose that we are given the algebraic expression $6x + 8$. Can you think of another way of writing this? (HINT: Factoring) What if I told you that this expression is the result of the Distributive Property of Multiplication Over Addition – that $6x + 8$ is the product of a binomial multiplied by a certain value. This is visualized by the partially detailed expression below.

$$\_\_\_ \, (\, \_\_\_ + \_\_\_ \,) = 6x + 8$$

To complete the expression, we must first determine the greatest factor that is shared between both terms. The *Greatest Common Factor* (GCF) between $6x$ and 8 is 2.

$$2 \, (\, \_\_\_ + \_\_\_ \,) = 6x + 8$$

Next, we must ask ourselves: 2 times *what* gives us $6x$ and 2 times *what* gives us 8? To determine this we may divide each term by 2 $\longrightarrow \frac{6x}{2} = 3x$ and $\frac{8}{2} = 4$.

$$2(3x + 4) = 6x + 8$$

Therefore, 2 and $3x+4$ are the factors of $6x+8$. A good practice is to always check our answers. We may do so using the Distributive Property of Multiplication Over Addition.

$$2(3x) + 2(4) = 6x + 8 \checkmark$$

Example 1:

Factor the trinomial: $27x^2 + 9x + 54$

$$9 \, (\, \_\_\_ + \_\_\_ + \_\_\_ \,) = 27x^2 + 9x + 54$$

$$\Downarrow$$

$$9(3x^2 + x + 6) = 27x^2 + 9x + 54$$

Solution: The greatest common factor is 9. We obtain our second factor, the trinomial inside the parentheses, by dividing each term by 9 ($\frac{27x^2}{9} = 3x^2$, $\frac{9x}{9} = x$ and $\frac{54}{9} = 6$).

Example 2:

Factor the trinomial: $14x^3 + 42x^2 + 70x$

$$14x\ (\ \_\_\_\ +\ \_\_\_\ +\ \_\_\_\ ) = 14x^3 + 42x^2 + 70x$$

$$\Downarrow$$

$$14x(x^2 + 3x + 5) = 14x^3 + 42x^2 + 70x$$

Solution: 14 is a common factor – but it is not the *greatest* common factor! Visible within every term is an x. Therefore, our expression is evenly divisible by $14x$. This is the greatest common factor. To obtain our second factor, the trinomial inside the parentheses, we must divide each term by $14x$ ($\frac{14x^3}{14x} = x^2$, $\frac{42x^2}{14x} = 3x$ and $\frac{70x}{14x} = 5x$).

Example 3:

Factor the trinomial: $-4xy^2 - 32xy - 20x^3y$

$$-4xy\ (\ \_\_\_\ +\ \_\_\_\ +\ \_\_\_\ ) = -4xy^2 - 32xy - 20x^3y$$

$$\Downarrow$$

$$-4xy(y + 8 + 5x^2) = -4xy^2 - 32xy - 20x^3y$$

Solution: The sign of each term is negative and each term is evenly divisble by $4xy$. Therefore, the greatest common factor is $-4xy$. To obtain our second factor, the trinomial inside the parentheses, we must divide each term by $-4xy$ ($\frac{-4xy^2}{-4xy} = y$, $\frac{-32xy}{-4xy} = 8$ and $\frac{-20x^3y}{-4xy} = 5x^2$).

3.2 – Factoring Quadratics

Factoring may be thought of as *undoing* multiplication. If we compare multiplication to wrapping a present, factoring is simply the unwrapping of the present to see what is inside! Now, as you may have noticed from section 2.5, when we multiply a binomial by a binomial our product is often a trinomial. In this section, I am going to show you how to unwrap a trinomial to see the two binomial factors inside.

Standard Form of a Quadratic Trinomial

$$ax^2 + bx + c$$

Recall from section 2.1 that a quadratic trinomial is another name for a 2^{nd} degree trinomial. In the standard form, our variable is x and our constants are represented by a, b, and c.

Factoring Quadratic Trinomials Where $a = 1$

$$(x + \underline{\;\boldsymbol{n_1}\;})(x + \underline{\;\boldsymbol{n_2}\;}) = x^2 + bx + c$$

To the left of the equal sign are our incomplete binomial factors. To the right of the equal sign is the standard form of a quadratic trinomial with a leading coefficient of 1 (a =1). To complete our binomial factors, we must find two numbers that satisfy the following equations:

$$(\;\underline{\;\boldsymbol{n_1}\;}\;) + (\;\underline{\;\boldsymbol{n_2}\;}\;) = b \qquad\qquad (\;\underline{\;\boldsymbol{n_1}\;}\;) \cdot (\;\underline{\;\boldsymbol{n_2}\;}\;) = c$$

These equations tell us that we must find two numbers that add to b *and* multiply to c. These two numbers will be used to complete our binomial factors.

$$(\;\underline{\;\boldsymbol{n_1}\;}\;) + (\;\underline{\;\boldsymbol{n_2}\;}\;) = \boldsymbol{b}$$

$$(\;\underline{\;\boldsymbol{n_1}\;}\;) \cdot (\;\underline{\;\boldsymbol{n_2}\;}\;) = \boldsymbol{c}$$

$$(x + \underline{\;\boldsymbol{n_1}\;})(x + \underline{\;\boldsymbol{n_2}\;}) = x^2 + \boldsymbol{b}x + \boldsymbol{c}$$

Example 1:

Factor the trinomial: $x^2 + 5x + 6$

Factors

$$\begin{array}{rcrl} 1 & \cdot & 6 = & 6 \\ \mathbf{2} & \cdot & \mathbf{3} = & \mathbf{6} \\ -1 & \cdot & -6 = & 6 \\ -2 & \cdot & -3 = & 6 \end{array}$$

$$(x + \underline{\quad})(x + \underline{\quad}) = x^2 + 5x + 6$$

$$2 + 3 = 5 \checkmark \qquad 2 \cdot 3 = 6 \checkmark$$

$$(x+2)(x+3) = x^2 + 5x + 6$$

Solution: We must find a pair of numbers that add to 5 and multiply to 6. To find this, we may list the factors of 6 and then determine which two add to 5. Both conditions are satisfied by 2 and 3. Therefore, 2 and 3 are used to complete our binomial factors. To check our work, we may multiply our factors together using Repeated Distribution or the Box Method.

Enrich Your Understanding: The order that we place our factors do not matter! The Commutative Property of Multiplication makes the following statement true: $(x+3)(x+2) = (x+2)(x+3)$.

Example 2:

Factor the trinomial: $x^2 - 14x + 24$

Factors

$$\begin{array}{rcrl} 1 & \cdot & 24 = & 24 \\ 2 & \cdot & 12 = & 24 \\ 3 & \cdot & 8 = & 24 \\ 4 & \cdot & 6 = & 24 \\ -1 & \cdot & -24 = & 24 \\ \mathbf{-2} & \cdot & \mathbf{-12} = & \mathbf{24} \\ -3 & \cdot & -8 = & 24 \\ -4 & \cdot & -6 = & 24 \end{array}$$

$$(x + \underline{\quad})(x + \underline{\quad}) = x^2 - 14x + 24$$

$$-2 + -12 = -14 \checkmark \qquad -2 \cdot -12 = 24 \checkmark$$

$$(x-2)(x-12) = x^2 - 14x + 24$$

Solution: We must find a pair of numbers that add to -14 and multiply to 24. To find this, we may list the factors of 24 and then determine which two add to -14. Both conditions are satisfied by -12 and -2. Therefore, -12 and -2 are used to complete our binomial factors.

Example 3:

Factor the trinomial: $x^2 + x - 72$

$$(x + \underline{\quad})(x + \underline{\quad}) = x^2 - 14x + 24$$

$$9 - 8 = 1 \checkmark \qquad\qquad 9 \cdot -8 = -72 \checkmark$$

$$(x + 9)(x - 8) = x^2 + x - 72$$

Solution: We must find a pair of numbers that add to 1 and multiply to -72. Recall, when a coefficient is hidden, it is inferred to be 1. Both conditions are satisfied by -8 and 9. Therefore, -8 and 9 are used to complete our binomial factors. In the previous examples, we listed out each pair of factors. This is not a necessary step. Sometimes we immediately recognize the correct numbers. However, if you do find yourself stuck, it is always helpful to include this step!

Factoring Quadratic Trinomials Where $a \neq 1$

$$(\underline{\quad}\, x + \underline{\quad})(\underline{\quad}\, x + \underline{\quad}) = ax^2 + bx + c$$

Factoring quadratic trinomials where $a \neq 1$ is similar to factoring quadratic trinomials where $a = 1$. As before, we must find two numbers that satisfy two conditions. The first condition is the same – we need two numbers that add to b. The second condition requires these two numbers to multiply to the product of a and c.

$$(\underline{\boldsymbol{p}}) + (\underline{\boldsymbol{q}}) = b \qquad\qquad (\underline{\boldsymbol{p}}) \cdot (\underline{\boldsymbol{q}}) = ac$$

Once two numbers are found that satisfy both conditions, we place them in a fraction over a and then simplify.

$$\begin{array}{ccc} \frac{\boldsymbol{p}}{a} & & \frac{\boldsymbol{q}}{a} \\ \downarrow & \text{Simplify} & \downarrow \\ \frac{n_1}{d_1} & & \frac{n_2}{d_2} \end{array}$$

Lastly, we place our simplified denominators as the coefficients and our simplified numerators as the accompanying term. Each fraction is dedicated to its own binomial factor (we must not mix-match numerators and denominators).

$$(\underline{\boldsymbol{d_1}}x + \underline{\boldsymbol{n_1}})(\underline{\boldsymbol{d_2}}x + \underline{\boldsymbol{n_2}}) = ax^2 + bx + c$$

n_1 = The reduced numerator of the 1^{st} fraction
d_1 = The reduced denominator of the 1^{st} fraction
n_2 = The reduced numerator of the 2^{nd} fraction
d_2 = The reduced denominator of the 2^{nd} fraction

Example 4:

Factor the trinomial: $2x^2 - 7x - 15$

$$(\ \underline{\quad}\ x + \ \underline{\quad}\ )(\ \underline{\quad}\ x + \ \underline{\quad}\ ) = 2x^2 - 7x - 15 \qquad -30$$

$$\begin{array}{ccc} -10 & + & 3 = -7 \checkmark \\ -10 & \cdot & 3 = -30 \checkmark \\ \downarrow & & \downarrow \\ \frac{-10}{2} & & \frac{3}{2} \\ \downarrow & & \downarrow \\ \frac{\mathbf{-5}}{\mathbf{1}} & & \frac{\mathbf{3}}{\mathbf{2}} \end{array}$$

$$(\underline{\mathbf{1}}x + \underline{\mathbf{-5}})(\underline{\mathbf{2}}x + \underline{\mathbf{3}}) = (x-5)(2x+3) = 2x^2 - 7x - 15$$

Solution: First, we find the product of a and c ($ac = 2 \cdot -15 = -30$). We must now find two numbers that add to -7 and multiply to -30. Both conditions are satisfied by -10 and 3. Each number is then used as a numerator for our fractions with $a = 2$ as the denominator. If possible, we simplify each fraction. The new denominators serve as our coefficients and the new numerators serve as our respective accompanying terms.

3.3 – Factoring by Grouping

In this section, we will discuss another method of factoring, called *grouping*. This method builds upon your knowledge of Greatest Common Factors (GCF), covered in section 3.1.

Factoring by grouping is used when there is no common factor amongst all of our terms, yet there is a common factor amongst some of our terms. To address this, we separate the terms within our polynomial into groups. Next, we factor out the GCF of each group. This reveals to us a common factor within our entire polynomial, allowing us to factor even further.

Example 1:

Factor the polynomial: $12x^3 + 18x^2 + 14x - 21$.

$$\begin{gathered} 12x^3 + 18x^2 + 14x - 21 \\ \Downarrow \\ (12x^3 + 18x^2) + (14x + 21) \\ \Downarrow \\ 6x^2(2x+3) + 7(2x+3) \\ \Downarrow \\ (2x+3)(6x^2+7) \end{gathered}$$

Solution: By observing our polynomial, we see that there is not a common factor amongst all of our terms – however, we do notice that there is a common factor of $6x^2$ amongst our first two terms and a common factor of 7 amongst our second two terms. With this in mind, we group our terms accordingly and then factor out their respective GCFs. The result of this is $6x^2(2x + 3) + 7(2x + 3)$. This reveals to us a common factor of $2x + 3$, allowing us to factor further.

Tip: For many – including myself – factoring out a binomial GCF is an uncomfortable task. To look past this, we may assign our binomial to a variable. For example, we may say, $2x+3 = p$.

$$\begin{gathered} 6x^2(2x+3) + 7(2x+3) \\ \Downarrow \\ 6x^2p + 7p \\ \Downarrow \\ p(6x^2+7) \\ \Downarrow \\ (2x+3)(6x^2+7) \end{gathered}$$

Example 2:

Factor the polynomial: $6xy^2 + 2x^2y - 9x - 27y$.

$$\begin{gathered} 6xy^2 + 2x^2y - 9x - 27y \\ \Downarrow \\ (6xy^2 + 2x^2y) + (-9x - 27y) \\ \Downarrow \\ 2xy(3y + x) - 9(x + 3y) \\ \Downarrow \\ (x + 3y)(2xy - 9) \end{gathered}$$

Solution: By observing our polynomial, we see that there is no common factor amongst all of our terms – however, we do notice that there is a common factor of $2xy$ amongst our first two terms and a common factor of -9 amongst our second two terms. With this in mind, we group our terms accordingly and then factor out their respective GCFs. When grouping, it is important to include the sign of each term within the parentheses, as $(-9x - 27y) \neq -(9x - 27y)$. The result of this is $2xy(3y + x) - 9(x + 3y)$. This reveals to us a common factor of $x + 3y$, allowing us to factor further. Recall, according the Commutative Property of Addition, $x + 3y = 3y + x$.

Example 3:

Factor the polynomial: $3x^2 - 12x - 5x + 20$.

$$\begin{gathered} 3x^2 - 12x - 5x + 20 \\ \Downarrow \\ (3x^2 - 12x) + (-5x + 20) \\ \Downarrow \\ 3x(x - 4) - 5(x - 4) \\ \Downarrow \\ (x - 4)(3x - 5) \end{gathered}$$

Solution: By observing our polynomial, we see that there is no common factor amongst all of our terms – however, we do notice that there is a common factor of $3x$ amongst our first two terms and a common factor of -5 amongst our second two terms. With this in mind, we group our terms accordingly and then factor out their respective GCFs. The result of this is $3x(x - 4) - 5(x - 4)$. This reveals to us a common factor of $x - 4$, allowing us to factor further.

Factoring Quadratic Trinomials by Grouping

In the previous section, we considered a method for factoring quadratic trinomials where a $\neq 1$. This method built upon the method used where $a = 1$. I will now provide you with an alternate method that requires factoring by grouping. You may use whichever method you prefer best!

We begin this method by performing the same initial steps as executed in section 3.2 – we find two numbers that add to b and multiply to ac. Once this is achieved, we rewrite our term with the coefficient of b as the sum, using the two numbers we discovered – i.e., if $p + q = b$ we write bx as $px + qx$. Think of this step as *uncombining* like terms. Now, we may proceed to factor by grouping.

Example 4:

Factor the trinomial: $2x^2 - 7x - 15$.

$$2x^2 - 7x - 15 \qquad\qquad -10 + 3 = -7 \checkmark$$

$$-30 \qquad\qquad -10 \cdot 3 = -30 \checkmark$$

$$\begin{gathered} 2x^2 - 7x - 15 \\ \Downarrow \\ 2x^2 - 10x + 3x - 15 \\ \Downarrow \\ (2x^2 - 10x) + (3x - 15) \\ \Downarrow \\ 2x(x - 5) + 3(x - 5) \\ \Downarrow \\ (x - 5)(2x + 3) \end{gathered}$$

Solution: First, we find the product of a and c ($ac = 2 \cdot -15 = -30$). Next, we find two numbers that add to -7 and multiply to -30. Both conditions are satisfied by -10 and 3. Next, since $-10 + 3 = -7$, we rewrite $-7x$ as $-10x + 3x$. Now, we may proceed to factor by grouping. By observing our polynomial, we see that there is not a common factor amongst all of our terms – however, we do notice that there is a common factor of $2x$ amongst our first two terms and a common factor of 3 amongst our second two terms. With this in mind, we group our terms accordingly and then factor out their respective GCFs. The result of this is $2x(x - 5) + 3(x - 5)$. This reveals to us a common factor of $x - 5$, allowing us to factor further.

Example 5:

Factor the trinomial: $6y^2 - 19y + 15$.

$$6y^2 - 19y + 15 \qquad -9 \;+\; -10 = -19\ \checkmark$$

$$90 \qquad -9 \;\cdot\; -10 = 90\ \checkmark$$

$$6y^2 - 19y + 15$$
$$\Downarrow$$
$$6y^2 - 10y - 9y + 15$$
$$\Downarrow$$
$$(6y^2 - 10y) + (-9y + 15)$$
$$\Downarrow$$
$$2y(3y - 5) - 3(3y - 5)$$
$$\Downarrow$$
$$(3y - 5)(2y - 3)$$

Solution: First, we find the product of a and c ($ac = 6 \cdot 15 = 90$). Next, we find two numbers that add to -19 and multiply to 90. Both conditions are satisfied by -10 and -9. Next, since $-10 + -19 = -19$, we rewrite $-19x$ as $-10x - 9x$. Now, we may proceed to factor by grouping. By observing our polynomial, we see that there is not a common factor amongst all of our terms – however, we do notice that there is a common factor of $2y$ amongst our first two terms and a common factor of -3 amongst our second two terms. With this in mind, we group our terms accordingly and then factor out their respective GCFs. Remember when grouping to include the sign of each term within the parentheses. The result of this is $2y(3y - 5) - 3(3y - 5)$. This reveals to us a common factor of $3y - 5$, allowing us to factor further.

3.4 – Difference of Two Perfect Squares

Whereas in most instances the product of two binomials is a trinomial, there exists a special case in which the product is also a binomial. This special case occurs given the following form:

$$(a+b)(a-b) = (a^2 - ab + ab - b^2) = (a^2 - b^2)$$

After we multiply the binomials together, we combine like terms ($-ab + ab = 0$). Remaining, are two terms – our *Difference of Two Perfect Squares* (DOTS). Our objective is to this recognize this pattern and then factor accordingly.

$$(a^2 - b^2) = (a+b)(a-b)$$

Example 1:

Factor the binomial: $x^2 - 4$.

$$(x^2 - 4) = (x^2 - 2^2) = (x-2)(x+2)$$

Solution: 4 is a perfect square ($4 = 2^2$). Since our binomial has the form $a^2 - b^2$, we can factor using DOTS ($a = x$ and $b = 2$).

Example 2:

Factor the binomial: $x^2 - 49$.

$$(x^2 - 49) = (x^2 - 7^2) = (x-7)(x+7)$$

Solution: 49 is a perfect square ($49 = 7^2$). Since our binomial has the form $a^2 - b^2$, we can factor using DOTS ($a = x$ and $b = 7$).

Example 3:

Factor the binomial: $x^2 + 64$.

Solution: Although 64 is a perfect square, we do not have the *difference* of two perfect squares. Our binomial must have the form $a^2 - b^2$ to factor using DOTS.

Example 4:

Factor the binomial: $64 - x^2$.

$$(64 - x^2) = (8^2 - x^2) = (8 - x)(8 + x)$$

Solution: 64 is a perfect square ($64 = 8^2$). Since our binomial has the form $a^2 - b^2$, we can factor using DOTS ($a = 8$ and $b = x$). Note, the order of the terms do not matter. The only requirement is that we have a difference of two perfect squares.

Example 5:

Factor the binomial: $-x^2 + 64$.

$$(-x^2 + 64) = (64 - x^2) = (8^2 - x^2) = (8 - x)(8 + x)$$

Solution: $-x^2 + 64$ *is* the difference of two perfect squares. We may use the Commutative Property of Addition to recognize this $(-x^2 + 64) = (64 + -x^2)$. Since our binomial has the form $a^2 - b^2$, we can factor using DOTS ($a = 8$ and $b = x$).

Example 6:

Factor the binomial: $25x^2 - 121$.

$$(25x^2 - 121) = ((5x)^2 - 11^2) = (5x - 11)(5x + 11)$$

Solution: $25x^2$ is a perfect square ($25x^2 = 5x \cdot 5x$) and 121 is also a perfect square ($121 = 11^2$). Since our binomial has the form $a^2 - b^2$, we can factor using DOTS ($a = 5x$ and $b = 11$).

Example 7:

Factor the binomial: $36x^2y^2 - 225z^2$.

$$(36x^2y^2 - 225z^2) = ((6xy)^2 - (15z)^2) = (6xy - 15z)(6xy + 15z)$$

Solution: $36x^2y^2$ is a perfect square ($25x^2 = 6xy \cdot 6xy$) and $225z^2$ is also a perfect square ($225z^2 = 15z \cdot 15z$). Since our binomial has the form $a^2 - b^2$, we can factor using DOTS ($a = 6xy$ and $b = 15z$).

3.5 – Factoring: Achieving Mastery

Now that you have studied multiple techniques for factoring, we will conclude this chapter with a variety of challenge problems. Keep in mind, some examples may require you to apply multiple techniques. These examples are designed to strengthen your identification skills.

Example 1:

Factor the trinomial: $2x^2 - 6x - 24$.

$\gg$ **Are we able to factor out a GCF?** Yessiree! Each term is evenly divisible by 2.

$$2x^2 - 8x - 24$$
$$\Downarrow$$
$$2(x^2 - 4x - 12)$$

$\gg$ **Do we have the difference of two perfect squares (DOTS)?** Psshh, quit playin'... we would need a binomial for that.

$\gg$ **Do we have a quadratic trinomial where a = 1?** Yes we do.

$$2(x + \_\_\_)(x + \_\_\_) = 2(x^2 - 4x - 12)$$

$$-6 + 2 = -4 \checkmark \qquad\qquad -6 \cdot 2 = -12 \checkmark$$

$$2(x - 6)(x + 2) = 2(x^2 - 4x - 12)$$

$\gg$ **Can we factor any further (with DOTS or GCF)?** Nope – we're done.

Note: The best way to ensure that you have completely factored a given expression is to ask yourself questions, as demonstrated above. After each step, list through the various methods of factoring. The expression is completely factored once no methods may be used. An excellent place to begin is by factoring out a GCF, if possible.

Example 2:

Factor the trinomial: $x^4 - 9x^2 + 18$.

$\gg$ **Are we able to factor out a GCF?** Nope.

$\gg$ **Do we have the difference of two perfect squares (DOTS)?** Nahh.

$\gg$ **Do we have a quadratic trinomial where a = 1?** Hmm...we do not – but we do have a trinomial of degree 4, where a = 1. I wonder if we are still able to factor it the same way. Let's try it!

$$x^4 - 13x^2 + 36$$
$$\Downarrow$$
$$(x^2 + \underline{\quad\;})(x^2 + \underline{\quad\;}) = x^4 - 13x^2 + 36$$

$$-9 + -4 = -13 \checkmark \qquad\qquad -9 \cdot -4 = 36 \checkmark$$

$$(x^2 - 9)(x^2 - 4) = x^4 - 13x^2 + 36$$

$\gg$ **Do we have the difference of two perfect squares (DOTS)?** Oh yes! We have it twice actually! $(x^2 - 9)$ and $(x^2 - 4)$ are both the difference of two perfect squares.

$$(x^2 - 9)(x^2 - 4) = x^4 - 13x^2 + 36$$
$$\Downarrow$$
$$(x - 3)(x + 3)(x - 2)(x + 2) = x^4 - 13x^2 + 36$$

$\gg$ **Can we factor any further (with DOTS or GCF)?** No, thank goodness!

Note: We may factor the trinomial $x^4 - 13x^2 + 36$ since the degree is double the middle term's exponent; 4 is twice as large as 2. To factor with this method, the degree must be double the middle term's exponent (e.g., $ax^4 + bx^2 + c$, $ax^6 + bx^3 + c$, $ax^8 + bx^4 + c$, $ax^{10} + bx^5 + c$...).

Enrich Your Understanding: When factoring a trinomial, the degree must be double the middle term's exponent since $x^n \cdot x^n = x^{n+n} = x^{2n}$ and $ax^n + bx^n = (a + b)x^n$. This is demonstrated below using the box method.

	x^2	-9
x^2		
-4		

$\Longrightarrow$

	x^2	-9
x^2	$\boldsymbol{x^4}$	$\boldsymbol{-9x^2}$
-4	$\boldsymbol{-4x^2}$	**36**

$\Longrightarrow x^4 - 4x^2 - 9x^2 + 36 \Longrightarrow x^4 - 13x^2 + 36$

Example 3:

Factor the trinomial: $36x^2y - 12x + 72xy - 24x$.

$\gg$ **Are we able to factor out a GCF?** Yes! Each term is evenly divisible by 12.

$$36x^2y - 12x + 72xy - 24$$
$$\Downarrow$$
$$12(3x^2y - x + 6xy - 2)$$

$\gg$ **Do we have the difference of two perfect squares (DOTS)?** No, like in example 1, we need a binomial for that.

$\gg$ **Do we have a quadratic trinomial where a = 1?** We have four terms and so, it is not a trinomial.

$\gg$ **Are we able to factor by grouping?** Let's try it! If we group our first two terms together and our last two terms together, we may factor out a GCF from each, providing us with the common factor of (3xy - 1).

$$12[(3x^2y - x) + (6xy - 2)]$$
$$\Downarrow$$
$$12[x(3xy - 1) + 2(3xy - 1)]$$
$$\Downarrow$$
$$12(3xy - 1)(x + 2)$$

$\gg$ **Can we factor any further (with DOTS or GCF)?** Nope. We're all done!

Example 4:

Factor the trinomial $16x^4y^8 - 81x^8y^4$.

$\gg$ **Are we able to factor out a GCF?** Yes! Each term is evenly divisble by x^4y^4.

$$16x^4y^8 - 81x^8y^4$$
$$\Downarrow$$
$$x^4y^4(16y^4 - 81x^4)$$

$\gg$ **Do we have the difference of two perfect squares (DOTS)?** Yupp! $(4y^2)^2 = 16y^4$ and $(9x^2)^2 = 81x^4$.

$$x^4y^4[(4y^2)^2 - (9x^2)^2]$$
$$\Downarrow$$
$$x^4y^4(4y^2 - 9x^2)(4y^2 + 9x^2)$$

$\gg$ **Do we have the difference of two perfect squares (DOTS)?** Wow, again! Okay! $(2y)^2 = 4y^2$ and $(3x)^2 = 9x^2$.

$$x^4y^4[(2y)^2 - (3x)^2](4y^2 + 9x^2)$$
$$\Downarrow$$
$$x^4y^4(2y - 3x)(2y + 3x)(4y^2 + 9x^2)$$

$\gg$ **Can we factor any further (with DOTS or GCF)?** Nope, we are all done!

Alternate Solution: Immediately, you may recognize that $16x^4y^8 - 81x^8y^4$ is the difference of two perfect squares and may be tempted to begin by factoring with DOTS. However, we also have a GCF of x^4y^4. I believe it is easier to factor out the GCF first. However, if you prefer, you may also begin factoring with DOTS. Below is the alternate work.

$$16x^4y^8 - 81x^8y^4$$
$$\Downarrow$$
$$(4x^2y^4 - 9x^4y^2)(4x^2y^4 + 9x^4y^2)$$
$$\Downarrow$$
$$(2xy^2 - 3x^2y)(2xy^2 + 3x^2y)(4x^2y^4 + 9x^4y^2)$$
$$\Downarrow$$
$$xy(2y - 3x) \cdot xy(2y + 3x) \cdot x^2y^2(4y^2 + 9x^2)$$
$$\Downarrow$$
$$x^4y^4(2y - 3x)(2y + 3x)(4y^2 + 9x^2)$$

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